JOURNAL OF COMBINATORIAL THEORY, Series A 71, 275–292 (1995)

# A Combinatorial Correspondence for Walks in Weyl Chambers

DAVID J. GRABINER*

*Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138*

*Communicated by the Managing Editors*

Received December 29, 1993

The $m \times m$ determinant of hyperbolic Bessel functions $\det |I_{a_i - b_j}(2x)|$ can be factored into two smaller determinants by elementary operations if $a_i = -a_{m+1-i}$ and $b_i = -b_{m+1-i}$. We give combinatorial interpretations for these determinants as exponential generating functions for walks which stay within Weyl chambers. We then use these to provide a combinatorial proof of the formulas by finding a sequence of reflections which give a correspondence between the walks enumerated on opposite sides of the equation. 

## 1. INTRODUCTION

Wilf [5] proved the following identity for Toeplitz determinants, where $t_l$ is an arbitrary sequence with $t_{-l} = t_l$.

$$\det_{2n \times 2n} |t_{i-j}| = \det_{n \times n} |t_{i-j} + t_{i+j-1}| \det_{n \times n} |t_{i-j} - t_{i+j-1}|. \tag{1}$$

The proof requires only elementary operations on the determinant. In the case where $t_l$ is the hyperbolic Bessel function $I_l(2x) = \sum_{k=0}^{\infty} x^{2k+l}/k!(k+l)!$, he provides a combinatorial interpretation for both sides.

Wilf also mentions that there is a similar formula in odd dimensions,

$$\det_{2n+1 \times 2n+1} |t_{i-j}| = 1/2 \det_{n+1 \times n+1} |t_{i-j} + t_{i+j-2}| \det_{n \times n} |t_{i-j} - t_{i+j}|, \tag{2}$$

but the combinatorial interpretation does not apply.

The same elementary operations can also be used to factor the more general class of determinants

$$\det_{m \times m} |t_{a_i - b_j}| \tag{3}$$

* Supported by an NSF graduate fellowship.



0097-3165/95 $12.00

for any decreasing sequences $a_i$ and $b_j$ with $a_i = -a_{m+1-i}$ and $b_j = -b_{m+1-j}$.

In the cases $t_l = I_l(2x)$ and $t_l = \binom{k}{k/2+l}$, we give combinatorial interpretations for these formulas, different from Wilf's interpretations, by showing that the determinants give the numbers of walks which stay within Weyl chambers. (For $t_l = I_l(2x)$, the determinant is the exponential generating function for walk-numbers.) In odd dimensions, both sides correspond to walks from $\vec{a}$ to $\vec{b}$ with the same permitted steps, but restricted to different Weyl chambers. In even dimensions, the $2n \times 2n$ determinant corresponds to a walk from $\vec{a}$ to $\vec{b}$, and the product of the two $n \times n$ determinants corresponds to a difference of two walks restricted to the same chamber, one from $\vec{a}$ to $\vec{b}$ and the other with destination $\vec{b}'$ obtained from $\vec{b}$ by interchanging $b_n$ and $b_{n+1}$.

In both cases, we use these interpretations to provide combinatorial proofs of the formulas by finding sequences of reflections which give a correspondence between walks which are enumerated on opposite sides of the equation.

In the case $t_l = I_l(2x)$, for (1) and (2), we also give a combinatorial interpretation of the determinants as generating functions for invariants in tensor powers of representations of Lie groups. When $a_i = (m+1)/2 - i$, (3) has a similar interpretation in terms of the decomposition of tensor powers.

## 2. Proofs by Elementary Row Operations

To prove the identity (1), or the more general identity

$$\det_{2n \times 2n} |t_{a_i - b_j}| = \det_{n \times n} |t_{a_i - b_j} + t_{a_i + b_j}| \det_{n \times n} |t_{a_i - b_j} - t_{a_i + b_j}| \tag{4}$$

with $a_i = -a_{2n+1-i}$, $b_j = -b_{2n+1-j}$, we do a sequence of elementary operations [5]. For each $i$ with $1 \leqslant i \leqslant n$, add each row $2n+1-i$ to row $i$; then, for $j$ with $1 \leqslant j \leqslant n$, subtract column $j$ from column $2n+i-j$. The resulting matrix is block lower triangular, with $n \times n$ blocks. The top right block is zero, so the determinant is the product of the diagonal blocks' determinants, which are the two factors on the right side of the equation.

Likewise, to prove (2), or the more general identity

$$\det_{2n+1 \times 2n+1} |t_{a_i - b_j}| = 1/2 \det_{n+1 \times n+1} |t_{a_i - b_j} + t_{a_i + b_j}| \det_{n \times n} |t_{a_i - b_j} - t_{a_i + b_j}|, \tag{5}$$

with $a_i = -a_{2n+2-i}$ for $1 \leqslant i \leqslant n$, $b_j = -b_{2n+2-j}$ for $1 \leqslant j \leqslant n+1$ (so $b_{n+1} = 0$), we do a similar sequence of elementary operations. For each $i$ with $1 \leqslant i \leqslant n$, add row $2n+2-i$ to row $i$, and double row $n+1$, thus doubling the determinant; then, for each $j$ with $1 \leqslant j \leqslant n$, subtract column

$j$ from column $2n+2-j$. Again, the top right $(n+1)\times n$ block of the resulting matrix is zero, so the the determinant is the product of the diagonal blocks' determinants.

In the case $t_l = I_l(2x)$, this gives

$$\det_{2n\times 2n} |I_{a_i-b_j}(2x)| = \det_{n\times n} |I_{a_i-b_j}(2x)+I_{a_i+b_j}(2x)| \det_{n\times n} |I_{a_i-b_j}(2x)-I_{a_i+b_j}(2x)|, \tag{6}$$

$$\det_{2n+1\times 2n+1} |I_{a_i-b_j}(2x)| = 1/2 \det_{n+1\times n+1} |I_{a_i-b_j}(2x)+I_{a_i+b_j}(2x)| \det_{n\times n} |I_{a_i-b_j}(2x)-I_{a_i+b_j}(2x)|. \tag{7}$$

Our combinatorial interpretations and proofs will apply to these identities, and to the analogous identities with $t_l = \binom{k}{k/2+l}$.

## 3. Walks in Weyl Chambers

### 3.1. *The Theorem of Gessel and Zeilberger*

To show that the determinants of Bessel functions are the exponential generating functions for walks in Weyl chambers, we use a result of Gessel and Zeilberger [1] which allows us to write the number of such constrained walks as a signed sum of the numbers of unconstrained walks with the same steps and lattice.

A walk-type is defined by a lattice $L$, a set $S$ of allowable steps between lattice points, and a polygonal cone $C$ to which the walks are confined.

We will assume $C$ is a *Weyl chamber*. That is, $L$, $S$, $C\subset\mathbb{R}^n$; $C$ is defined by a system of simple roots $\Delta\subset\mathbb{R}^n$ as

$$C=\{\vec{x}\in\mathbb{R}^n \mid (\alpha,\vec{x})\geqslant 0 \text{ for all } \alpha\in\Delta\}; \tag{8}$$

the orthogonal reflections $r_\alpha\colon \vec{x}\mapsto\vec{x}-2((\alpha,\vec{x})/(\alpha,\alpha))\alpha$ preserve $L$ and $S$; and the $r_\alpha$ generate a finite group $W$ of linear transformations, the *Weyl group*.

Definition. A walk-type $(L, S, C)$ is *reflectable* if the following equivalent conditions hold:

1. Any step $s\in S$ from any lattice point in the interior of $C$ will not exit $C$.

2. For each simple root $\alpha_i$, there is a real number $k_i$ such that $(\alpha_i, s)=\pm k_i$ or $0$ for all steps $s\in S$ and $(\alpha_i,\lambda)$ is an integer multiple of $k_i$ for all $\lambda\in L$.

The reflectability condition guarantees that a walk cannot exit $C$ without landing on a wall of $C$ at some step.

In a reflectable walk problem, we want to compute $b_{\eta\lambda, k}$, the number of walks from $\eta$ to $\lambda$ of length $k$ which stay in the interior of a Weyl chamber, in terms of $c_{\gamma k}$, the number of unconstrained walks with the same steps which go from the origin to $\gamma$. The fundamental result of Gessel and Zeilberger [1] is:

THEOREM 1. *If the walk from $\eta$ to $\lambda$ is reflectable, then*

$$b_{\eta\lambda, k} = \sum_{w \in W} \operatorname{sgn}(w)\, c_{\lambda - w(\eta), k,}. \tag{9}$$

*Proof.* Every walk from any $w(\eta)$ to $\lambda$ which does touch at least one wall has some last step $j$ at which it touches a wall; let the wall be the hyperplane perpendicular to $\alpha_i$, choosing the largest $i$ if there are several choices [3]. Reflect all steps of the walk up to step $j$ across that hyperplane; the resulting walk is a walk from $w_{\alpha_i} w(\eta)$ to $\lambda$ which also touches wall $i$ at step $j$. This clearly gives a pairing of walks, and since $w_{\alpha_i}$ has sign $-1$, these two walks cancel out in (9). The only walks which do not cancel in these pairs are the walks which stay within the Weyl chamber, and since $w(\eta)$ is inside the Weyl chamber only if $w$ is the identity, this is the desired number of walks.

### 3.2. *Unconstrained Walks*

In two important cases, $c_{\gamma k}$, or its exponential generating function $h_\gamma(x) = \sum_{k=0}^{\infty} c_{\gamma k} x^k/k!$, is easy to compute.

If the steps $S$ are $\pm e_i$, the positive and negative coordinate directions, then

$$h_\gamma(x) = \sum_{k=0}^{\infty} \prod_{i=1}^{n} \exp(x(u_i + u_i^{-1}))\,|_{\vec{u}^\gamma},$$

where $|_{\vec{u}^\gamma}$ denotes the coefficient of $\vec{u}^\gamma$ in the polynomial in $u$.

These functions are products of hyperbolic Bessel functions, since

$$\begin{aligned}
\exp(x(u+u^{-1})) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=-k}^{k} \binom{k}{j} u^{k-2j} \\
&= \sum_{m=-\infty}^{\infty} u^m \sum_{k=0}^{\infty} \frac{x^k}{k!} \binom{k}{(k+m)/2} \\
&= \sum_{m=-\infty}^{\infty} u^m \sum_{k=0}^{\infty} \frac{x^{2k+m}}{k!(k+m)!} \\
&= \sum_{m=-\infty}^{\infty} u^m I_m(2x),
\end{aligned}$$

Thus, in this case, the exponential generating function for the unconstrained walks is

$$h_\gamma(x)=\prod_{i=1}^{n} I_{\gamma_i}(2x). \tag{10}$$

Note that if we add a step of 0 to the walk, this multiplies the exponential generating function by $e^x$.

If, instead, the steps $S$ are the diagonals $\pm\frac{1}{2}e_i$, then the generating function for a single step is

$$\chi(\vec{u})=\prod_{i=1}^{n} (u_i^{-1/2}+u_i^{1/2}),$$

and thus the coefficient of $\vec{u}^\gamma$ in the $k$th power of this product is

$$c_{\gamma k}=\prod_{i=1}^{n} \binom{k}{k/2+\gamma_i}. \tag{11}$$

### 3.3. *The Formulas for Constrained Walks*

The Weyl groups we will consider on $n$-dimensional lattices are $A_{n-1}$, the symmetric group $S_n$ of coordinate permutations; $B_n$, the hyperoctahedral group, which consists of all coordinate permutations with sign changes; and $D_n$, the even hyperoctahedral group, which consists of all coordinate permutations with an even number of sign changes. The principal Weyl chamber for $A_{n-1}$ is given by $x_1>x_2>\cdots>x_n$; for $B_n$, it is given by $x_1>\cdots>x_n>0$; for $D_n$, it is given by $x_1>\cdots>x_{n-1}>x_n$, $x_{n-1}>-x_n$.

The lattice $\mathbb{Z}^n$ gives a reflectable walk with all three groups for the steps $\pm e_i$; the lattice $D_n^*$, containing $\mathbb{Z}^n$ and $(\mathbb{Z}+\frac{1}{2})^n$, gives a reflectable walk in all cases except for the steps $\pm e_i$ with the group $B_n$.

In all of these cases, we can write the sum over the permutations in the Weyl group as a determinant, with separate terms in each entry of the matrices for the choices of sign changes for $w\in B_n$ or $D_n$. The formulas are derived in [2].

Since we had an exponential generating function for the unconstrained walks with steps $\pm e_i$, we will also get an exponential generating function for the constrained walks. For these walks, the functions below are only valid if $\eta_i\equiv\lambda_i \pmod 1$, so that the indices of the Bessel functions are integers; otherwise, there are obviously no walks. With steps $\pm e_i$, for $A_n$, we have

$$g_{\eta\lambda}(x)=\det_{n\times n} |I_{\lambda_i-\eta_j}(2x)|. \tag{12}$$

For $B_n$ (with $\eta$, $\lambda \in \mathbb{Z}^n$), we have

$$g_{\eta\lambda}(x) = \det_{n\times n} |I_{\lambda_i - \eta_j}(2x) - I_{\lambda_i + \eta_j}(2x)|. \tag{13}$$

For $D_n$, we have

$$g_{\eta\lambda}(x) = \tfrac{1}{2}[\det_{n\times n} |I_{\lambda_i - \eta_j}(2x) - I_{\lambda_i + \eta_j}(2x)| + \det_{n\times n} |I_{\lambda_i - \eta_j}(2x) + I_{\lambda_i + \eta_j}(2x)|]. \tag{14}$$

In the formula (14), if we let $\lambda'$ be obtained from $\lambda$ by changing the sign of $\lambda_n$, this will change the sign of the first term but preserve the second term. Thus the first term alone, with no factor of $\frac{1}{2}$, is $g_{\eta\lambda}(x) - g_{\eta\lambda'}(x)$, and the second term alone is $g_{\eta\lambda}(x) + g_{\eta\lambda'}(x)$ . If $\lambda_n = 0$ or $\eta_n = 0$, the first term is zero, so $g_{\eta\lambda}(x)$ is the second term alone, with the factor of $\frac{1}{2}$.

For the diagonal walks with steps $\pm\frac{1}{2}e_1 \cdots \pm\frac{1}{2}e_n$ on the lattice $D_n^*$, we have the values of $c_{\gamma k}$ for the unconstrained walks, so we will have the values of $b_{\eta\lambda, k}$ for the constrained walks. For $A_{n-1}$, we have

$$b_{\eta\lambda, k} = \det_{n\times n} \left| \binom{k}{\frac{k}{2} + \lambda_i - \eta_j} \right|. \tag{15}$$

For $B_n$, we have

$$b_{\eta\lambda, k} = \det_{n\times n} \left| \binom{k}{\frac{k}{2} + \lambda_i - \eta_j} - \binom{k}{\frac{k}{2} + \lambda_i + \eta_j} \right|. \tag{16}$$

For $D_n$, we have

$$b_{\eta\lambda, k} = \frac{1}{2}\left[ \det_{n\times n} \left| \binom{k}{\frac{k}{2} + \lambda_i - \eta_j} - \binom{k}{\frac{k}{2} + \lambda_i + \eta_j} \right| \right.$$
$$\left. + \det_{n\times n} \left| \binom{k}{\frac{k}{2} + \lambda_i - \eta_j} + \binom{k}{\frac{k}{2} + \lambda_i + \eta_j} \right| \right]. \tag{17}$$

Again, in this formula, we can get the first term alone as a difference of walk-numbers $b_{\eta\lambda, k} - b_{\eta\lambda', k}$ and the second term alone as $b_{\eta\lambda, k} + b_{\eta\lambda', k}$.

### 3.4. *The Walk Identities*

Note that the determinants on the left side of (6) and (7) are already in the form (12), while the determinants on the right side are either of the form (13) or the terms in (14). Thus, if we let

$$\vec{a} = (\eta_1, ..., \eta_n, -\eta_n, ..., -\eta_1)$$

in (6), and similarly for $\vec{b}$ and $\lambda$, we see that the exponential generating function for walks from $\vec{a}$ to $\vec{b}$ with Weyl group $A_{2n-1}$ is equal to the product of the two exponential generating functions for $D_n$,

$$[g_{\eta\lambda}(x) - g_{\eta\lambda'}(x)][g_{\eta\lambda}(x) + g_{\eta\lambda'}(x)] = g_{\eta\lambda}(x)^2 - g_{\eta\lambda'}(x)^2. \tag{18}$$

Now, the number of walks from $\eta$ to $\lambda$ which stay within the $D_n$ Weyl chamber is also the number of walks from $\bar{\eta} = (-\eta_n, ..., -\eta_1)$ to $\bar{\lambda} = (-\lambda_n, ..., -\lambda_1)$ which stay within the $D_n$ Weyl chamber containing them; this chamber is defined by $x_1 > x_2 > \cdots > x_n$, $-x_1 > x_2$. Thus we can write this product as

$$g_{\eta\lambda}(x)\, g_{\bar{\eta}\bar{\lambda}}(x) - g_{\eta\lambda'}(x)\, g_{\bar{\eta}\bar{\lambda}'}(x)$$

Finally, we write these products of two walks with Weyl group $D_n$ as single walks in $2n$ dimensions with Weyl group $D_n \times D_n$. Note that $(\eta, \bar{\eta}) = \vec{a}$, $(\lambda, \bar{\lambda}) = \vec{b}$, and $(\lambda', \bar{\lambda}') = \vec{b}'$, where $\vec{b}'$ is obtained from $\vec{b}$ by switching $b_n$ and $b_{n+1} = -b_n$. Thus we have the following result.

THEOREM 2. *If $\vec{a}$ and $\vec{b}$ satisfy the antisymmetry conditions $a_i = -a_{2n+1-i}$, $b_j = -b_{2n+1-j}$, then the number of walks with steps $\pm e_i$ from $\vec{a}$ to $\vec{b}$ which stay within the $A_{2n-1}$ Weyl chamber is equal to the number of walks from $\vec{a}$ to $\vec{b}$ which stay within the $D_n \times D_n$ Weyl chamber, minus the number of walks from $\vec{a}$ to $\vec{b}'$ which stay within the $D_n \times D_n$ Weyl chamber.*

Similarly, in odd dimensions, we let

$$\vec{a} = (\eta_1, ..., \eta_n, a_{n+1}, -\eta_n, ...)$$

in (7), and likewise for $\vec{b}$, which must have $b_{n+1} = 0$. Also let $\hat{\eta} = (\eta, a_{n+1})$, $\hat{\lambda} = (\lambda, 0)$. The left side of (7) is the product of $g_{\hat{\eta}\hat{\lambda}}(x)$ with Weyl group $D_{n+1}$ (since $\hat{\lambda}_{n+1} = 0$) and $g_{\eta\lambda}(x)$ with Weyl group $B_n$.

Again, we can convert $\eta$ and $\lambda$ to $\bar{\eta}$ and $\bar{\lambda}$ in the $B_n$ determinant, and write the product of walks on $\mathbb{Z}^{n+1}$ and $\mathbb{Z}^n$ as a single walk on $\mathbb{Z}^{2n+1}$. This gives the following result.

THEOREM 3. *If $\vec{a}$ and $\vec{b}$ satisfy the antisymmetry conditions $a_i = -a_{2n+2-i}$ for $1 \leqslant i \leqslant n$, $b_j = -b_{2n+2-j}$ for $1 \leqslant i \leqslant n$, with $b_{n+1} = 0$, then the* *number of walks with steps $\pm e_i$ from $\vec{a}$ to $\vec{b}$ which stay within the $A_{2n}$ Weyl chamber is equal to the number of walks from $\vec{a}$ to $\vec{b}$ which stay within the $D_{n+1} \times B_n$ Weyl chamber.*

This theorem also holds if $a_{n+1}=0$ and $b_{n+1} \neq 0$, since we can run the walk in the opposite direction; it does not hold if both are nonzero.

By a similar argument, we see that both theorems also hold for the walk with steps $\pm\frac{1}{2}e_1 \cdots \pm\frac{1}{2}e_n$.

For either walk-type, both theorems, and particularly Theorem 3, call out for a combinatorial proof by giving a direct correspondence between the walks. We do have combinatorial proofs, given in Section 5, but these proofs require a long sequence of reflections to get the correspondence between different walks.

## 4. Decomposition of Tensor Powers

An important application of the determinant formulas for walks is the decomposition of tensor powers. Using the Weyl Character and Integration Formulas, we can express the decomposition of tensor powers as a sum of unconstrained walks. Then, using Theorem 1, we can prove the following result [2].

Theorem 4. *Let $V$ be a finite-dimensional representation of a connected reductive Lie group $G$. Let $C$ be a Weyl chamber, $S$ the set of weights of $V$, and $L$ some lattice containing $S$ and $\rho$, the half-sum of the positive roots.*

*If $(L, S, C)$ defines a reflectable walk-type, then the number $b_{\rho, \rho+\mu, k}$ of walks with $k$ steps from $\rho$ to $\rho+\mu$ which stay strictly within the principal Weyl chamber is equal to the multiplicity $a_{\mu, k}$ of the irreducible with highest weight $\mu$ in the $k$th tensor power of $V$. Likewise, if $f_\mu$ is the exponential generating function $\sum_{k=0}^{\infty} a_{\mu, k} x^k/k!$, it is equal to the exponential generating function $g_{\rho, \rho+\mu}$ for walks.*

If $\mu=0$, $V_\mu$ is the trivial representation, and thus $a_{\mu, k}$ is the dimension of the space of invariant tensors of rank $k$.

We apply this theorem to the representations of Lie groups which have weights $\pm x_i$. For $GL_m$, this is the direct sum of the defining and the dual representations, the Weyl group is $A_{m-1}$, $\rho$ is $((m-1)/2, (m-3)/2, ..., -(m-1)/2)$, and we use (12). For $Sp_{2n}$, it is the defining representation, the Weyl group is $C_n = B_n$, and $\rho$ is $(n, n-1, ..., 1)$, and we use (13). For $SO_{2n}$, it is the defining representation, the Weyl group is $D_n$, and $\rho$ is $(n-1, n-2, ..., 0)$, so the first term in (14) will be zero and we use only the second term in (14).

For the Lie group $SO_{2n+1}$, the Weyl group is $B_n$, and $\rho$ is $((n-1)/2, (n-3)/2, ..., 1/2)$. The standard representation $\chi$ has weights $\pm x_i$ and 0, so the virtual representation $\chi - 1$ has weights $\pm x_i$. The constrained walk with Weyl chamber $B_n$ does not satisfy the reflectability conditions; however, we can express the sum of unconstrained walks as a difference of two constrained walks with Weyl group $D_n$, taking the walks from $\rho$ to $\rho + \mu$ and subtracting the walks to $(\rho + u)'$, which is obtained from $\rho + \mu$ by reversing the sign of the last coordinate [2]. That is, we have

$$f_\mu(x) = g_{\rho, \rho+\mu}(x) - g_{\rho, (\rho+\mu)'}(x), \tag{19}$$

and the generating function is the first term of (14). For the tensor powers of the virtual representation $1 - \chi$, the generating function is $f_\mu(-x)$; that is, the sign of every term corresponding to a walk of odd length changes. If $\sum \mu_i$ is even, then every walk from $\rho$ to $\rho + \mu$ is of even length, and every walk from $\rho$ to $(\rho + \mu)'$ is of odd length, so we will get the sum of the walks to those two points; if $\sum \mu_i$ is odd, we will get the negative of the sum. The generating function will thus be the second term in (14) (with a minus sign if appropriate); that is,

$$f_\mu(-x) = (-1)^{\Sigma \mu_i} g_{\rho, \rho+\mu}(x) + g_{\rho, (\rho+\mu)'}(x). \tag{20}$$

Thus, for the product group $SO_{2n+1} \times SO_{2n+1}$, with Weyl group $D_n \times D_n$, and representation the direct sum of the two virtual representations above, the generating function $f_\mu(x)$ will be the product of these two functions, corresponding to walks with steps that belong to either of the two walks. As in the proof of Theorem 3, this product is the exponential generating function for constrained walks from $(\rho, \bar{\rho})$ to $(\rho + \mu, \overline{\rho + \mu})$, minus the exponential generating function for walks from from $(\rho, \bar{\rho})$ to $((\rho + \mu)', \overline{(\rho + \mu)'})$, where $\bar{\lambda} = (-\lambda_n, ..., -\lambda_1)$ as before. The product representation is

$$((\chi - 1), 1) + (1, (1 - \chi)) = (\chi, 1) - (1, \chi)$$

Thus, from Theorem 2, or just by checking that the resulting determinants match those in (6), we have the following result.

THEOREM 5. *The multiplicity of the representation with highest weight $(\mu, \bar{\mu})$ in the $k$th tensor power of the direct sum of the defining and dual representations for $GL_{2n}$ is the same as the multiplicity of the representation with highest weight $(\mu, \mu)$ in the $k$th tensor power of the virtual representation $(\chi, 1) - (1, \chi)$ of $SO_{2n+1} \times SO_{2n+1}$.*

In particular, if $\mu=0$, we find that the invariants in the tensor powers of these representations of $GL_{2n}$ and $SO_{2n+1}\times SO_{2n+1}$ have the same dimension.

The odd-dimensional case is similar. Again, as in the proof of Theorem 3, we write the $Sp_{2n}$ walk as a walk from $\bar{\rho}$ to $\overline{\rho+\mu}$. We can then apply Theorem 3, or check that the determinants we get match those in (7), to get the following result.

THEOREM 6. *The multiplicity of the representation with highest weight* $(\mu_1, ..., \mu_n, \mu_{n+1}, -\mu_n, ..., -\mu_1)$ *in the kth tensor power of the direct sum of the defining and dual representations for* $GL_{2n+1}$ *is the same as the multiplicity of the representation with highest weight* $(\mu_1, ..., \mu_{n+1}; \mu_1, ..., \mu_n)$ *in the kth tensor power of the representation* $(\chi, 1)+(1, \chi)$ *of* $SO_{2n+2}\times Sp_{2n}$.

Again, when $\mu=0$, this says that the invariants in the tensor powers of these representations have the same dimensions.

## 5. COMBINATORIAL PROOF OF THE DETERMINANT IDENTITIES

We will prove the identities (6) and (7) (and the analogous identities with determinants instead of Bessel functions) by looking at the signed terms corresponding to individual walks enumerated in Theorem 1, which can be seen to correspond to the individual terms in the determinants in (6) and (7). The combinatorial proof of Theorem 1 shows that all walks which touch a wall of the Weyl chamber cancel out, leaving only the walks which stay in the Weyl chamber. For each term, we then find a set of reflections which pair the walks contributing to it with the walks contributing to another term of the same sign with the same value on the other side, or of the opposite sign with the same value on the same side. We then use the involution principle to get an actual correspondence between the unconstrained walks from this pairing.

We will prove the correspondence for walks which go from $\vec{a}=(\eta_1, ..., \eta_n, -\eta_n, ..., -\eta_1)$ to $\vec{b}=(\lambda_1, ..., \lambda_n, -\lambda_n, ..., -\lambda_1)$ in $2n$ dimensions, and for walks between $\vec{a}=(\eta_1, ..., \eta_n, \eta_{n+1}, -\eta_n, ..., -\eta_1)$ and $\vec{b}=(\lambda_1, ..., \lambda_n, 0, -\lambda_n, ..., -\lambda_1)$ in $2n+1$ dimensions. We let $m=2n$ or $2n+1$, as appropriate.

By reversing the direction of the walks in Theorem 1, we see that the number of constrained walks from $\vec{a}$ to $\vec{b}$ is equal to the signed sum of the number of constrained walks from $\vec{a}$ to $\vec{w}(b)$, for $w$ in the Weyl group. We will use the theorem in this form.

5.1. *Array Notation*

When we discuss the term for walks from $\vec{a}$ to $\vec{w}(b)$, or to any $\vec{c}=(c_1, ..., c_m)$ whose coordinates are either 0 or $\pm\lambda_i$, we will represent this walk by an array with two rows, with the first row containing the first $n$ or $n+1$ coordinates of $\vec{c}$, and the last row containing the last $n$ in reverse order. Thus two coordinates of $\vec{c}$ will be together in column $i$ if the walk takes one coordinate from $\eta_i$ to the coordinate in the top row, and the other coordinate from $-\eta_i$ to the coordinate in the bottom row. Thus, the walk from $\vec{a}$ to $\vec{b}$ itself has the corresponding array form

$$\begin{array}{ccccc} \lambda_1 & \lambda_2 & \cdots & \lambda_n & (0) \\ -\lambda_1 & -\lambda_2 & \cdots & -\lambda_n & \end{array}$$

where the 0 in parentheses appears only in dimension $2n+1$.

In the following discussion, we will say that a walk or array is *valid on the left side* if it corresponds to a term from the left side of (6) or (7); that is, if it is a walk from $a$ to $w(b)$ for some $w \in A_{2n-1}$ or $A_{2n}$. We likewise define walks and arrays which are valid on the right side. The discussion in the following section illustrates the correspondence between these terms and the walks; a formal proof is given in [2]. We will implicitly use this correspondence in the proof of the identities.

5.2. *Signs of Arrays*

Each array which corresponds to a term on either side in (6) or (7) has some sign. We will start by defining the sign of such an array to be the sign of the term in the equation.

The sign of a term from the left side of (6) or (7) is the sign of the permutation that was applied to the $b_i$; this permutation could also be applied to the array form. We get one term for every permutation.

In even dimensions, the terms on the right side of (6) come from the application of $D_n \times D_n$ to $\vec{b}=(\lambda, \bar{\lambda})$, with sign corresponding to the element of $D_n \times D_n$; or its application top $\vec{b'}=(\lambda', \bar{\lambda}')$ obtained by switching $\lambda_n$ and $-\lambda_n$ as before, with an additional negative sign. $D_n \times D_n$ acts on the array form by permuting the terms in each row, and then changing the sign of an even number of terms in each row. Since $\vec{b}$ itself has a bottom row which is entirely negative, while the bottom row of $\vec{b'}$ has one positive sign, the sign of the resulting term is the product of the signs of the permutations on the two rows, multiplied by $-1$ if there are an odd number of positive signs in the bottom row. We get one term for every signed permutation of the entries in each row in which the number of negative signs in the top row and the number of positive signs in the bottom row have the same parity.

In odd dimensions, the terms on the right side of (7) come from the application of $D_{n+1} \times B_n$ to $\vec{b}$. Here, $D_{n+1}$ acts on the top row by permuting the elements, and changing an even number of signs; $B_n$ acts on the bottom row by permuting the elements, and changing any number of signs. The sign of the term will thus be the product of the signs of the permutations on the two rows, multiplied by $-1$ if there are an odd number of positive signs in the bottom row. We get one term for every signed permutation, since we can make the number of sign changes in the top row even by changing the sign of the zero in the top row.

In both odd and even dimensions, we can give a simpler definition of the sign of the term corresponding to an array. First change the signs of every element of the top row which duplicates an element of the bottom row; this corresponds to an even element of $D_n$ or $D_{n+1}$. (If $m = 2n$, this is an even number of sign changes; if $m = 2n + 1$, we can change the sign of the zero in the top row to make it an even number of sign changes.) The sign of the term for the original array is the sign of the permutation of the resulting array. This procedure works because switching $\lambda_i$ and $-\lambda_i$, which is a transposition, changes the sign; after all the entries are in the correct row for their signs, we can apply permutations to each row, which are elements of $D_n \times D_n$ or $D_{n+1} \times B_n$, to get the original order. Here is an example.

$$\operatorname{sgn}\begin{pmatrix} -\lambda_1 & -\lambda_3 & -\lambda_2 \\ -\lambda_2 & \lambda_1 & -\lambda_3 \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} -\lambda_1 & \lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_1 & -\lambda_3 \end{pmatrix}$$

This permutation in $S_6$ is $(\lambda_1, -\lambda_1, -\lambda_2)(\lambda_2, \lambda_3)(-\lambda_3)$, which is odd, so the sign of the original array is also $-1$.

### 5.3. *Star Notation*

We now generalize the set of arrays to allow any array which has two entries whose absolute value is $\lambda_i$ for each $i$, and a zero if the dimension is $2n + 1$, provided that no two equal entries are in the same row. We require that a star be placed on one element of each pair of identical entries, and define the sign of the array to be the sign of the permutation in $S_m$ which results after the sign of every starred element is changed. Such an array is a valid array on the left side if there are no two equal entries anywhere (and thus no stars); it is valid on the right side if not two entries in the same row have the same absolute value, and all the stars are in the top row. (In odd dimensions, the zero must be in the top row for this to occur, since there are $n + 1$ entries in the top row and only $n$ different nonzero absolute values.)

Using our previous example,

$$\operatorname{sgn}\begin{pmatrix} -\lambda_1 & -\lambda_3^* & -\lambda_2^* \\ -\lambda_2 & \lambda_1 & -\lambda_3 \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} -\lambda_1 & \lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_1 & -\lambda_3 \end{pmatrix} = -1.$$

Note that moving a single star changes the sign; for example, if the $-\lambda_3$ in the bottom row had been starred instead, the sign would be $+1$. Also note that, if there is a zero, a star on it does not affect the sign, since $-0=0$.

### 5.4. *The Cross-Switch*

If we reflect an entire walk across the hyperplane $x_i + x_{m+1-i} = 0$ (which contains the starting point $\vec{a}$), we interchange positive steps in direction $i$ with negative steps in direction $m+1-i$, and vice versa. If the original walk had destination $\vec{c} = (c_1, ..., c_m)$, the new walk has a destination $\vec{c}'$ whose $i$th coordinate is $-c_{m+1-i}$ and whose $(m+1-i)$th coordinate is $-c_i$. Thus this reflection gives a natural combinatorial correspondence between walks to $\vec{c}$ and $\vec{c}'$.

The effect on the array notation is also easy to describe, so we give a name to the corresponding operation on arrays.

DEFINITION. In an array, we define a *cross-switch* of column $i$ to be the operation which interchanges the two elements of the column, and changes both signs.

Note that the new array may not have the same sign, depending on where we choose to place the stars after the cross-switch. Our pairing of walks will match arrays (and thus walks) which correspond by a set of cross-switches, and we will keep track of the stars to guarantee that the signs cancel out correctly.

### 5.5 *Cycle Notation*

If we do a cross-switch which moves an entry $\pm\lambda_j$, it will also affect the other entry in the same column. In addition, it may affect the star (or lack of one) on the other entry with absolute value $\lambda_j$. Thus we will draw a graph on the array, joining all entries which are in the same column, and all entries which have the same absolute value. (If the two entries in a column have the same absolute value, draw a double edge connecting them.)

In $2n$ dimensions, every node in this graph has degree 2, so the graph is a union of cycles. In $2n+1$ dimensions, the entry in the last column has no vertical edge, and the zero has no edge joining it to another entry with the same absolute value; thus the graph will contain a chain with an even number of edges (possibly 0) and these entries as endpoints.

When we cross-switch a column, this will switch the location of two entries in the cycle or chain which were joined by a vertical edge. If one entry was previously joined to the other entry with the same absolute value by a horizontal edge, it will now be joined by a diagonal edge, and conversely.

We now speak of cycles or chains as valid on the left or right side if they satisfy the same conditions as for full arrays (no two equal entries in a left-side cycle; all stars in the rop row and no two entries with equal absolute values in the same row in a right-side cycle), and thus are allowed to appear in arrays which are valid on the left or right side.

### 5.6. *Orientation of Cycles*

We now choose an origin and an orientation for each cycle. The choice is arbitrary, except that the origin must be in the top row; to make it unique, let the origin be the top entry of the first column in each cycle, and choose the orientation so that the edge going away from this entry is the edge connecting it to the other entry of the same absolute value. If there is a chain, let the entry in the last column be the origin; this gives an orientation for the chain.

One result of this choice of origins is that the stars in a cycle or chain from the right side will all be an even distance away from the origin. We will choose the cross-switches, and the new placement of stars after the cross-switches, to preserve this property.

### 5.7. *Corresponding Walks on the Same Side*

If an array which is valid on the right side has any cycle (not the chain) containing an odd number of stars, we will pair it with another array, also valid on the right side, of opposite sign, by doing a set of cross-switches which do not preserve the sign.

In the array, choose the cycle containing an odd number of stars which contains the first possible column. Consider the array with every column of this cycle cross-switched. Carry the stars down the columns; since the sign of both entries of the same absolute value will be changed at the same time, each duplicated pair will still contain one star. For example,

$$\begin{matrix} 3^* & 2 & 1 \\ -1 & 3 & -2 \end{matrix} \to \begin{matrix} 1 & -3 & 2 \\ -3^* & -2 & -1 \end{matrix}.$$

Assume that there are $j$ columns in the cycle. Then the operation on the array will all starred entries corrected consists of transposing the two entries in each column ($j$ transpositions) and then switching each pair $x$ and $-x$ (another $j$ transpositions). Thus these two cycles have the same sign.

The new cycle has all of its stars, an odd number, in the bottom row. To make it valid for the right side, we need to move the stars to the top row; since moving one star changes the sign of an array, the new array will have an opposite sign to the original array.

Since they are equivalent by cross-switches, the walks will cancel out; it is also clear that this operation is its own inverse, so we have now found a pairing which cancels out all walks on the right side in which some cycle has an odd number of stars.

### 5.8. *Preserving the Sign with a Cross-Switch*

In the rest of the correspondence, we can choose the positioning of stars to guarantee that our cross-switches preserve the sign of an array, provided that we do not cross-switch the column containing the origin. When we cross-switch a column, this affects the need for stars in two pairs of numbers. We will add or remove the stars of the two affected entries which are an even distance from the origin of the cycle. Thus, for example, if the 2 in the third column is an even distance from the origin:

$$\begin{array}{ccc} 1 & -2 & 2 \\ & 1^* & \end{array} \rightarrow \begin{array}{ccc} 1 & -1 & 2^* \\ & 2 & \end{array}$$

Because of our choice of the positions of stars, exactly one of the two changed stars is outside the column that was cross-switched. Thus, if we change the sign of all starred entries and then consider the resulting permutations, they will differ by a 3-cycle (the cycle is $(-1, 2, -2)$ above). The entry in the cross-switched column whose star was changed (the 1 above, becoming a $-1$ after the sign change) moves to the other spot in the column. The entry that was there moves to the spot outside the column where the star was changed (above, the $-2$ in the left-hand diagram moves to the third column of the right-hand diagram, since there is now a $2^*$ there). And the entry that was there moves to the place inside the column where the other entry with the same absolute value is now, which is where the first entry was (above, the 2 in the third column goes to the unstarred 2 in the right-hand diagram). Since this is a 3-cycle, the two diagrams have the same sign.

If the zero in the chain is involved in such a switch, it will be the entry in the column whose star is switched, since the chain has an even number of edges. Thus the sign will still be preserved, by the above argument. The zero will acquire a star, but this star has no effect on the sign, since $-0 = 0$; it can thus be dropped.

### 5.9. *Corresponding Walks on Opposite Sides*

The remainder of the correspondence will match left-side arrays with right-side arrays of the same sign, rather than matching pairs of arrays with different signs. We have already eliminated all arrays in which a cycle (not the chain) has an odd number of stars.

If a cycle or chain came from the left side, follow the chain or cycle away from its origin, and cross-switch every column preceded by an odd number of horizontal edges. Thus, if two columns were joined by a horizontal edge, one of them will be cross-switched and the other one will not be; if they were joined by a diagonal edge, either both or neither will be cross-switched. For example (the origin here is the 3 in the first column, so the second and third columns are switched because they are preceded by the edge joining the 3 to the $-3$):

$$\begin{array}{ccc} 3 & 2 & -3 \\ -1 & 1 & -2 \end{array} \rightarrow \begin{array}{ccc} 3^* & -1^* & 2 \\ -1 & -2 & 3 \end{array}$$

If this is a cycle, it has the same number of entries in each row and thus the same number of horizontal edges in each row, so the number of horizontal edges is even. Thus the edge joining the last column to the column containing the origin will not cause a problem; this edge was diagonal before the switches if and only if all of the horizontal edges preceded the last column, causing it not be switched. In the chain, there is no such edge, so it will not cause a problem.

These cross-switches will create new stars. Since the new cycle or chain has no horizontal edges and the new stars are an even distance from the origin, they will all be in the same row as the origin, which is the top row, so the resulting cycle will be valid on the right side. The stars will be created only in the first of a pair of columns, exactly one of which was switched (since this is the entry an even distance from the origin); that is, in the columns that were the tail end of horizontal edges.

Conversely, to convert a valid right-side cycle with an even number of stars, or a valid right-side chain, to left-side form, we cross-switch every column which is preceded by an odd number of stars, not counting a star in the column itself. This will remove all the stars, since we are cross-switching one of every two adjacent columns if the edge connecting them has a star; in a cycle, the last column will be cross-switched if and only if it contains a star, and this will remove the star. For an example, consider the diagram above, with the arrow reversed.

It is clear that this is a bijection, since horizontal edges are converted to edges with stars at their tail ends by the cross-switching, and vice versa. Thus, since all of the cross-switches preserve sign, we can perform this operation on all cycles in a valid left-side array to get a valid right-side array which cancels it out in (6) or (7). This correspondence covers all of the remaining valid arrays, completing our combinatorial proof.

5.10. *A Bijection Between the Constrained Walks*

We can use this combinatorial proof to produce a bijection between the constrained walks in Theorem 2 or 3, using the *involution principle*, which is described in [4, Section 2.6]. Start with a constrained walk which appears in the theorem. This constrained walk is also an unconstrained walk. Now, repeat the following process as long as possible:

1. Starting with an unconstrained walk to some destination, replace it by the unconstrained walk to the corresponding destination obtained by the above process. (Recall that the cross-switches give a bijection of walks, not merely of destinations.) This is always possible.

2. If the current walk hits any walls of the Weyl chamber, replace it by the walk which cancels it out in Theorem 1. This is possible unless the walk satisfies the conditions for a constrained walk.

Since each step is uniquely reversible and there are only a finite number of possible walks, this process must eventually terminate, providing a pairing between constrained walks which are valid on one side or the other of (6) or (7).

If we transfer all terms of (6) or (7) to the same side of the equation, then every use of either step above changes the sign of the term to which the walk contributes. Since the process both begins and ends with Step 1, the initial and final walks must contribute to terms of opposite sign. Thus we have a pairing between the constrained walks contributing to terms of opposite sign in Theorem 2 or 3. This gives the desired bijection.

### 5.11. *Examples*

We will now work out the complete correspondence for the simplest non-trivial case, walks of length 4 which both start and end at $(1, 0, -1)$.

First, we find the correspondence of destinations of unconstrained walks. There are six $A_2$-images of $(1, 0, -1)$; they are $(1, 0, -1)$, $(0, -1, 1)$, and $(-1, 1, 0)$ with positive sign, and $(1, -1, 0)$, $(-1, 0, 1)$, and $(0, 1, -1)$ with negative sign. The eight $D_2 \times B_1$-images are $(\pm 1, 0, -1)$ and $(0, \pm 1, 1)$ with positive sign, and $(\pm 1, 0, 1)$ and $(0, \pm 1, -1)$ with negative sign.

In cycle notation, the destinations with 0 as the second entry (which goes into the last column) have a chain with no edges containing just the 0, and the two $\pm 1$'s in a cycle; the others have a chain connecting the second entry to the other $\pm 1$, and then to the 0. Thus there are no horizontal edges in the cycles of any $A_2$-image except for $(1, -1, 0)$ and $(-1, 1, 0)$; the correspondence thus takes the other four destinations to themselves, and it can be checked that the signs match.

For $(1, -1, 0)$, with negative sign for $A_2$, we have the array $\left(\begin{smallmatrix} 1 & -1 \\ 0 & \end{smallmatrix}\right)$, which gives a chain with origin at the $-1$. The first column is preceded by an odd number of horizontal edges (the edge joining 1 to $-1$), so it must be cross-switched; the cross-switch gives the array $\left(\begin{smallmatrix} 0^* & -1^* \\ -1 & \end{smallmatrix}\right)$, corresponding to the destination $(0, -1, -1)$. This also has negative sign for $D_2 \times B_1$.

Going in the other direction, the first column is preceded by an odd number of stars in the chain (the star on the $-1$), so it must be cross-switched, giving the corresponding destination $(1, -1, 0)$. Likewise, $(-1, 1, 0)$ for $A_2$ corresponds to $(0, 1, 1)$ for $D_2 \times B_1$.

There are two remaining $D_2 \times B_1$ terms, $(1, 0, 1)$ and $(-1, 0, -1)$; they have opposite signs. In array notation, $(1, 0, 1)$ is $\left(\begin{smallmatrix}1^* & 0\\ 1 & \end{smallmatrix}\right)$. The first column is the only 2-cycle with an odd number of stars, so we cross-switch it and move the star back to the top row to get $\left(\begin{smallmatrix}-1^* & 0\\ -1 & \end{smallmatrix}\right)$. This corresponds to the destination $(-1, 0, -1)$, which does have the opposite sign as required. We now have the complete correspondence for the terms in (7).

Now, we build the correspondence of walks. There are 12 constrained walks with Weyl group $A_2$, and 11 of them are also valid with Weyl group $D_2 \times B_1$, so the correspondence takes them to themselves.

The twelfth walk is $3-, 2-, 2+, 3+$, where $i+$ and $i-$ denote positive and negative steps in the $i$th coordinate direction. The destination $(1, 0, -1)$ for $A_2$ is matched with the same destination, and thus the same walk for $D_2 \times B_1$. This walk is first on a $D_2 \times B_1$ wall after the second step, when it has $x_1 = -x_2$, so we reflect it across tha wall from that point on to get $3-, 2-, 1-, 3+$. This unconstrained walk has destination $(0, -1, -1)$, so the corresponding destination is the walk to $(1, -1, 0)$ on the $A_2$ side; cross-switching coordinate directions 1 and 3 (reflecting the walk across the plane $x_1 + x_3 = 0$) takes the walk to $1+, 2-, 3+, 1-$. This walk is first on an $A_2$ wall after the second step, when it has $x_2 = x_3$, so we reflect across that wall from the point on to get $1+, 2-, 2+, 1-$. The destination of this walk is $(1, 0, -1)$, which is paired with itself on the $D_2 \times B_1$ side in the correspondence of destinations, so the walk doesn't change. And the walk is a valid $D_2 \times B_1$ walk, so our process terminates; the $A_2$ walk $3-, 2-, 2+, 3+$ pairs with the $D_2 \times B_1$ walk $1+, 2-, 2+, 1-$.

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